# Duality of Sensor Network Design Models for Parameter Estimation

**Miguel J. Bagajewicz and Mabel C. Sánchez**
School of Chemical Engineering and Materials Science, University of Oklahoma, The Energy Center,
Rm T-355, Norman, OK 73019

Plant-performance evaluation relies on the estimation of specific variables, which could be model parameters or performance indicators. Thus, the availability of reliable process knowledge is essential to peformance evaluation. This information is obtained through monitoring and data reconciliation only when an adequate set of instruments has been located at the right places. The measurement arrangement should guarantee the observability and precision of the variables involved in the estimation scheme. Furthermore, the assessment of cost-optimal measurement structures for performance estimation is a challenging issue for complex plants.

Several authors have addressed the problem of selecting measurement structures to determine accurate parameter values. Furthermore, several articles have appeared in the literature to design sensor networks for steady-state process. The designs satisfied different purposes, such as observability, precision, cost, reliability, and robustness. A survey of the state of the art can be found in Bagajewicz (1997). Among these strategies, we are concerned with maximum-precision models and minimum-cost models for parameter estimation. In this regard, there seems to be confusion in the literature as of which model is best for design. In addition, there is no study performing comparisons or giving recommendations.

In this article a brief overview of the minimum-cost approach (Bagajewicz, 1997) and maximum-precision models are presented first. In the next section a MINLP generalized maximum-precision model for multiple-parameter estimation is proposed and its advantages over other existing techniques are highlighted. Following that, the duality between the maximum-precision model and the minimum-cost model is established. Finally, illustrative examples are solved.

Correspondence concerning this article should be addressed to M. J. Bagajewicz.
Current address for M. C. Sánchez: Planta Piloto de Ingeniería Química (UNS-CONICET), C.C. 717, Camino La Carrindanga Km 7, (8000) Bahía Blanca, Argentina.

## Minimum-Cost Model

Consider a process whose steady-state operation is described by the nonlinear algebraic system

$$f(z) = 0, \tag{1}$$

where $z$ contains the vector $w$ of measurable state variables and the vector of unmeasurable process parameters $\theta$

$$z = \begin{bmatrix} w \\ \theta \end{bmatrix}. \tag{2}$$

Due to cost or technical feasibility, not all state variables in $z$ are measured, so let $q$ be a vector of binary variables such that $q_i = 1$, if $w_i$ is measured, and zero otherwise. Consider a set $M_p$ of unmeasured parameters that it is desired to estimate. The selection of instruments such that the cost is minimized and accuracy constraints on parameters are satisfied, is the solution of the following optimization problem:

$$\begin{aligned} &\text{Min} \sum_{i \epsilon M_i} c_i q_i \\ &\text{s.t.} \\ &\sigma_j(q) \le \sigma_j^* \qquad \forall j \epsilon M_p \\ &q_i = 0,1 \qquad \forall i \epsilon M_1, \end{aligned} \tag{3}$$

where $\sigma_j(q)$ represents the standard deviation of the estimated value of parameter $\theta_j$ obtained after data reconciliation, and $\sigma_j^*(q)$ are the threshold values. This formulation is only a special case of the more general problem presented in Bagajewicz (1997). For a selected set of instruments represented by the vector $q$, the standard deviation $\sigma_j(q)$ may be estimated by first performing an observability analysis of the unmeasured variables included in the linearized model. If this analysis indicates that all parameters in $M_p$ are observable, then the set of proposed instruments represented by $q$ is feasible. Then, precision of the estimated parameters can be undertaken using data-reconciliation procedures. Incidentally, one can also impose precision constraints on measured variables.

## Maximum-Precision Models

Maximum-precision models include all those models developed for sensor-network design that contain a measure of the estimation quality of parameters or state variables in the objective function. In some models precision is maximized, in others error estimates are minimized. For steady-state systems with only random measurement noise, Kretsovalis and Mah (1987) developed a combinatorial strategy to incorporate measurements to an observable system. They minimized an objective function that makes a weighted average between the cost of the measurements and the precision obtained. In practice, however, it is very difficult to assess the weights. Later on, Madron and Veverka (1992) proposed to design sensor networks in order to minimize the mean square error of the variables by considering hardware redundancy and minimizing a weighted sum of the estimated parameters' precisions. A version without hardware redundancy can be solved using the concept of a maximum-cost spanning tree.

In a recent article, Alheritiere et al. (1997) evaluated the contribution of process data to performance measures and proposed a scheme for a cost-benefit analysis of resource reallocation. The article proposes that the optimal redistribution of fixed resources to the different sensors of an existing plant be obtained in order to maximize the precision of a parameter estimate. The authors propose as a starting point the following nonlinear optimization problem:

$$\begin{aligned} &\min_{c_i} \ \sigma_\theta \\ &\text{s.t.} \\ &\sum_{i \in M_1} c_i(\sigma_m) = c_T \\ &c_i^L \le c_i \le c_i^U \qquad \forall i \in M_1, \end{aligned} \tag{4}$$

where $c_T$ is the total resource allocated to all sensors and $\sigma_m$ is the vector of measurement standard deviations. The standard deviation of the parameter estimate is obtained without applying data reconciliation. *This type of NLP is not appropriate from the point of view of sensor network design and/or upgrade because:* (1) the model is confined to the estimation of one parameter; (2) the continuous representation of variables leads to nondiscrete values for the number of sensors; (3) different sets of measurements can lead to the estimation of the same parameter, but the selected procedure chooses one *a priori*; and (4) if a set of measurements leading to the estimation of the parameter is redundant, a smaller variance for each variable can be obtained using data reconciliation. Therefore fewer measurements can accomplish the same parameter variance. If data-reconciliation procedures are not applied, then, in the presence of redundant equations, there are several different sets of nonredundant measurements that can be used to obtain the parameter. In the absence of information regarding how good each of these sets is, one can only resort to using the set that will provide the best precision, or resort to averaging. No option is better than straight data reconciliation, however, especially because data reconciliation is always accompanied by gross error detection.

## Generalized Maximum-Precision Model

In this section a generalized model for maximum precision is proposed. The model considers the minimization of a weighted sum of the parameters' precision constrained by cost.

$$\begin{aligned} &\min \sum_{j \in M_p} a_j \sigma_j^2(q) \\ &\text{s.t.} \\ &\sum_{i \in M_1} c_i q_i \le c_T \\ &q_i = 0,1 \qquad \forall i \in M_1. \end{aligned} \tag{5}$$

When hardware redundancy is used, then an upper bound on the cost for each instrument (or on number of instruments, if they are all equal) can be imposed. Lower bounds on cost are not needed.

*The main advantages of the generalized model over other existing techniques are*: (1) it can provide a design for multiple-parameter estimation; (2) since binary variables are used, more realistic results can be obtained in accordance with the discrete nature of sensors; (3) it takes into account redundancy and all possible forms of obtaining the parameters; and (4) it has previous models (Madron and Veverka, 1992; Madron, 1992) as a particular case. Furthermore problem 4 is also a particular case of problem 5. Indeed, the objective function in problem 4 is the objective function of problem 5 written for only one parameter. In addition, when $q$ is *a priori* selected, as in problem 4, then the problem becomes NLP. Finally, if the cost constraint in problem 4 is replaced by an inequality ($\sum_{i \in M_1} c_i(\sigma_m) \le c_T$), this constraint will be binding, simply because the cost is a monotone function of precision. This, however, may not happen if all the upper bounds in costs have been reached, but this would make problem 4 infeasible. The main disadvantage of this model is that there is no criterion for picking the weights in the objective function, although practitioners may find some particular criteria for specific cases.

## Duality of Sensor Network Models

In this section a mathematical connection between the generalized maximum-precision model, given by problem 5, and the minimum-cost model, given by problem 3, is presented. We first modify the generalized maximum-precision model by simply adding upper bounds on the precision of the parameters; these are trivial constraints if these upper bounds are properly selected. Similarly, the minimum-cost model is modified by the addition of a trivial constraint consisting of the weighted average sum of existing precision constraints

($\sigma_j(q) \leq \sigma_j^*$). The resulting models are:

$$\left.\begin{array}{l} \text{Min} \sum_{j \in M_p} a_j \sigma_j^2(q) \\ \text{s.t.} \\ \sum_{i \in M_1} c_i q_i \leq c_T \\ \sigma_j(q) \leq \sigma_j^*(q) \qquad \forall j \in M_p \\ q_i = 0,1 \qquad \forall i \in M_1 \end{array}\right\} \tag{6}$$

$$\left.\begin{array}{l} \text{Min} \sum_{i \in M_1} c_i q_i \\ \text{s.t.} \\ \sum_{j \in M_p} a_j \sigma_j^2 \leq \sum_{j \in M_p} a_j (\sigma_j^*)^2 \\ \sigma_j(q) \leq \sigma_j^* \qquad \forall j \in M_p \\ q_i = 0,1 \qquad \forall i \in M_1 \end{array}\right\}. \tag{7}$$

We are now in a position to show that the minimum-cost model is the dual of the maximum-precision model in the Tuy sense. This duality was established in general by Tuy (1987), and a shortened version is reproduced in the Appendix. Thus, in our case:

$$\left\{\begin{array}{l} c_T = \alpha \leq \text{Min} f(q) = \text{Min} \sum_{i \in M_1} c_i q_i \\ \text{s.t.} \\ g(q) = -\sum_{j \in M_p} a_j \sigma_j^2 \geq -\sum_{j \in M_p} a_j (\sigma_j^*)^2 = \beta \\ \sigma_j(q) \leq \sigma_j^* \qquad \forall j \in M_p \\ q_i = 0,1 \qquad \forall i \in M_1 \end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{l} \beta = -\sum_{j \in M_p} a_j (\sigma_j^*)^2 \geq \text{Max}\, g(q) = \text{Min} \sum_{j \in M_p} a_j \sigma_j^2 \\ \text{s.t.} \\ f(q) = \sum_{i \in M_1} c_i q_i \leq c_T = \alpha \\ \sigma_j(q) \leq \sigma_j^* \qquad \forall j \in M_p \\ q_i = 0,1 \qquad \forall i \in M_1 \end{array}\right\}. \tag{8}$$

This implies that (1) if the constraint on cost $c_T$ for the maximum-precision model is smaller than the optimum cost obtained from the minimum-cost model, then the weighted sum of variances in the maximum-precision model does not reach its maximum possible value given by the weighted sum of bounds; (2) if the optimum value of the weighted sum of variances obtained in the maximum-precision model is smaller than the weighted sum of bounds, then the solution of the minimum-cost model is larger than the maximum cost used in the maximum-precision models; (3) when the minimum cost obtained from the minimum-cost model is used as an upper bound on cost in the maximum-precision model, then the weighted sum of variances is equal to the weighted sum of bounds. This also implies that the constraint on weighted averages is binding in the minimum-cost model.

In other words, *the solution of one problem is one solution of the other and vice versa*.

Finally, bounds on precision come from model−plant mismatch considerations and its effect on the economics of the process. For example, in a recent article, Loeblein and Perkins (1998) suggest ways to quantify the impact of model−plant mismatch in the economics of on-line optimization.

**Table 1. Set of Possible Instruments for the Flash Tank**

| Measured Variable | Instrument Cost | Instrument Standard Deviation | Measured Variable | Instrument Cost | Instrument Standard Deviation |
|---|---|---|---|---|---|
| $F_1$ | 250 | 3.0 | $y_{32}$ | 700 | 0.01 |
| $y_{11}$ | 700 | 0.015 | $F_3$ | 300 | 1.418 |
| $y_{21}$ | 700 | 0.015 | $y_{13}$ | 800 | 0.01 |
| $y_{31}$ | 700 | 0.015 | $y_{23}$ | 800 | 0.01 |
| $F_2$ | 250 | 1.515 | $y_{33}$ | 800 | 0.01 |
| $y_{12}$ | 700 | 0.01 | $P$ | 100 | 14 |
| $y_{22}$ | 700 | 0.01 | | | |

**Table 2. Results for the Minimum Cost Model**

| Case # | $\sigma^*$ | $\sigma$ | $c_{\min}$ | Optimal Set |
|---|---|---|---|---|
| 1 | 0.05 | 0.00865 | 1600 | $y_{12}$, $y_{33}$, $P$ |
| 2 | 0.007 | 0.005896 | 1750 | $F_1$, $y_{12}$, $y_{22}$, $P$ |
| 3 | 0.006 | 0.005896 | 1750 | $F_1$, $y_{12}$, $y_{22}$, $P$ |
| 4 | 0.0058 | 0.00551 | 2300 | $y_{22}$, $y_{32}$, $y_{13}$, $P$ |
| 5 | 0.005 | 0.004982 | 2900 | $F_1$, $F_2$, $y_{22}$, $y_{13}$, $y_{23}$, $P$ |

## Illustrative Examples

To illustrate this part, consider the flash-tank example taken from Van Winkle (1967) and used by Alheritiere (1998). In this case, the vaporization efficiency coefficient $\eta$ is estimated in terms of measurements that are obtained installing instruments from the set presented in Table 1.

The minimum-cost model was solved using different bounds for the standard deviation of the parameter. Optimal total cost of instrumentation, parameter standard deviation, and the optimal set of instruments for each run are provided in Table 2. Table 3 presents results for the maximum-precision model for different bounds on standard deviation of the parameter and total cost. As the results indicate, instrumentation cost increases with precision requirements. Sometimes no feasible set is available when cost constraints are imposed. Two examples (Cases 1 and 2) are included where the minimum cost for the minimum-cost model is considered as a bound for the maximum-precision model, both models were run with the same bound for the parameter standard deviation, and they produce the same results.

**Table 3. Results for the Maximum Precision Model**

| Case # | $\sigma^*$ | $c_T^*$ | $\sigma$ | $c$ | Set |
|---|---|---|---|---|---|
| 1 | 0.0500 | 1600 | 0.00865 | 1600 | $y_{12}$, $y_{33}$, $P$ |
| 2 | 0.0070 | 1750 | 0.005896 | 1750 | $F_1$, $y_{12}$, $y_{22}$, $P$ |
| 3 | 0.0058 | 3600 | 0.0048 | 3600 | $F_1, y_{21}, F_2, y_{22}, y_{13}, y_{23}, P$ |
| 4 | 0.0055 | 1750 | — | — | — |
| 5 | 0.0050 | 2500 | — | — | — |
| 6 | 0.0050 | 3000 | 0.004982 | 2950 | $F_2$, $y_{22}$, $F_3$, $y_{13}$, $y_{23}$, $P$ |

**Table 4. Results for the Minimum Cost Model**

| Case # | $\sigma^*_{\eta_1}$ | $\sigma^*_{\eta_2}$ | $\sigma^*_{\eta_3}$ | $\sigma_{\eta_1}$ | $\sigma_{\eta_2}$ | $\sigma_{\eta_3}$ | $c_{\min}$ | Optimal Set |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.1 | 0.2 | 0.09 | 0.0990 | 0.0265 | 0.0520 | 3100 | $y_{22}, y_{32}, y_{23}, y_{33}, P$ |
| 2 | 0.07 | 0.09 | 0.09 | 0.0660 | 0.0288 | 0.0486 | 3900 | $y_{12}, y_{32}, y_{13}, y_{23}, y_{33}, P$ |
| 3 | 0.06 | 0.06 | 0.6 | 0.0572 | 0.0218 | 0.0425 | 4600 | $y_{12}, y_{22}, y_{32}, y_{13}, y_{23}, y_{33}, P$ |
| 4 | 0.04 | 0.05 | 0.05 | — | — | — | — | — |

We now illustrate the case of multiple-parameter estimation. Consider the same flash example, with a different equilibrium correction parameter for each component, that is $y_{i3} = \eta_i y_{i2} P_i(\text{sat})/P$. The minimum-cost model was run using the maximum variances indicated in Table 4, where optimal solutions are presented. The optimal cost of instrumentation increases with the precision of the parameters' estimates, but, for the last case, no set of available instruments can fulfill precision requirements.

Optimal designs obtained by applying maximum-precision models are presented in Tables 5 and 6. For cases 1 to 4, all weights in the objective function are considered equal to one. In case 1, bounds on the standard deviation are removed. In cases 2 to 4, higher costs and parameters' precision are allowed leading to a lower weighted sum of parameter variances.

For cases 5 to 8, the weights in the objective function values are [5 1000 1]. The same optimal set is obtained for cases 1 and 6, 3 and 7 of Table 5. They have the same feasible region and differ only in weight values. Higher precision is required in case 8, and an unfeasible situation is presented in case 9. Case 2 and cases 3 and 7 of Tables 4 and 5 are examples of duality.

From the preceding examples, some conclusions and recommendations can be made. If bounds on standard deviation of the parameters are available, then minimum-cost models are a better alternative, because the selection of weights for the objective function can be avoided. If in turn, bounds are not available, maximum-precision models constrained only by cost can be used, using, for example, all weights equal to one. Different weights may be selected that reflect the relative importance of the precision of the parameters. Instead of this, however, after the maximum-precision model is run with proposed weights, one can use the minimum-cost model and perform a more meaningful sensitivity analysis in terms of cost, by using as bounds on precision, values suggested by the result of the maximum-precision model.

**Table 5. Results for the Maximum Precision Model**

| Case # | $\sigma^*_{\eta_1}$ | $\sigma^*_{\eta_2}$ | $\sigma^*_{\eta_3}$ | $c_T$ | $\sigma_{\eta_1}$ | $\sigma_{\eta_2}$ | $\sigma_{\eta_3}$ |
|---|---|---|---|---|---|---|---|
| 1 | — | — | — | 3100 | 0.0990 | 0.0265 | 0.05209 |
| 2 | 0.07 | 0.09 | 0.09 | 3900 | 0.0660 | 0.0288 | 0.0486 |
| 3 | 0.06 | 0.06 | 0.06 | 4600 | 0.0573 | 0.02180 | 0.04258 |
| 4 | 0.06 | 0.06 | 0.06 | 7000 | 0.0571 | 0.02179 | 0.04254 |
| 5 | 0.5 | 0.5 | 0.5 | 4000 | 0.07008 | 0.026572 | 0.73572 |
| 6 | 0.5 | 0.5 | 0.5 | 3100 | 0.0990 | 0.0265 | 0.05209 |
| 7 | 0.06 | 0.06 | 0.06 | 4600 | 0.0573 | 0.02180 | 0.04258 |
| 8 | 0.06 | 0.03 | 0.05 | 6500 | 0.05713 | 0.02178 | 0.04256 |
| 9 | 0.05 | 0.03 | 0.05 | 9000 | — | — | — |

**Table 6. Results for the Maximum Precision Model**

| Case | $c$ | $\sum_i \sigma^2_{\eta_i}$ | Optimal Set |
|---|---|---|---|
| 1 | 3100 | 0.01326 | $y_{22}, y_{32}, y_{23}, y_{33}, P$ |
| 2 | 3900 | 0.00750 | $y_{12}, y_{32}, y_{13}, y_{23}, y_{33}, P$ |
| 3 | 4600 | 0.00557 | $y_{12}, y_{22}, y_{32}, y_{13}, y_{23}, y_{33}, P$ |
| 4 | 6800 | 0.00555 | $F_1, y_{11}, y_{31}, F_2, y_{12}, y_{22}, y_{32}, F_3, y_{13}, y_{23}, y_{33}, P$ |
| 5 | 3600 | 0.736044 | $F_1, F_2, y_{12}, y_{22}, y_{13}, y_{23}, P$ |
| 6 | 3100 | 0.757837 | $y_{22}, y_{32}, y_{23}, y_{33}, P$ |
| 7 | 4600 | 0.493668 | $y_{12}, y_{22}, y_{32}, y_{13}, y_{23}, y_{33}, P$ |
| 8 | 6500 | 0.492814 | $F_1, y_{11}, y_{21}, F_2, y_{12}, y_{22}, y_{32}, y_{13}, y_{23}, y_{33}, P$ |
| 9 | — | — | — |

## Conclusions

The duality between a generalized maximum-precision model and the minimum-cost model for sensor network design have been established. Some discussion is provided about the convenience of applying these methods when the cost or precision bound information is missing.

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## Appendix: Tuy Duality Theorem

Consider the problems $P_\beta$: Inf$\{f(x)$: $x \in V$, $g(x) \geq \beta\}$ and $Q_a$: Sup$\{g(x)$: $x \in V$, $f(x) \leq \alpha\}$, where $V$ is an arbitrary set in $R^n$; $f$: $R^n \to R$ and $g$: $R^n \to R$ are two arbitrary functions; and $\alpha$ and $\beta$ are two real numbers. Then if the solutions of both $Q_a$ and $P_\beta$ are bounded, $\alpha = \min\ P_\beta \Leftrightarrow \beta = \max\ Q_a$.

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